Bulletin of Networking, Computing, Systems, and Software – www.bncss.org, ISSN 2186-5140
Volume 14, Number 1, pages 77–80, January 2025

Reservoir Computing Techniques Using Tensor Networks

1st Shinji Sato
Engineering Department
The University of Electro-Communications
Tokyo, Japan
shinji.sato@uec.ac.jp

2nd Daiki Sasaki
Engineering Department
The University of Electro-Communications
Tokyo, Japan
dsasaki@uec.ac.jp

3rd Chih-Chieh Chen
Grid Inc.
Tokyo, Japan
chen.chih.chieh@gridsolar.jp

4th Kodai Shiba
Grid Inc.
Tokyo, Japan
shiba.kodai@gridsolar.jp

5th Tomah Sogabe
Engineering Department
The University of Electro-Communications
Tokyo, Japan
sogabe@uec.ac.jp

***Abstract*—Reservoir computing (RC), traditionally based on echo state networks and liquid state machines, shows great potential in modeling dynamic time-series data like weather and astronomical predictions. However, these frameworks are not suitable for quantum dynamics-based RC. Tensor networks (TNs) are well suited for modeling quantum dynamics because they can efficiently model quantum information and entanglement. In this work, we propose a novel randomized TN-based RC scheme, demonstrating its validity through various case studies. Our results show superior performance compared to traditional ESN models, laying the groundwork for further exploration of quantum reservoir computing.**

***Index Terms*—Reservoir Computing (RC), Tensor Networks (TN), Quantum Dynamics.**

I. Introduction

A. Background

Reservoir computing (RC) has established itself as a powerful approach for learning dynamic temporal data, primarily through frameworks like echo state networks (ESNs) and liquid state machines, where only the output weights are trained, and internal weights remain fixed [1]–[13]. Traditional RC, however, struggles with quantum dynamics due to limitations in representing high-dimensional quantum information and entanglement [14]–[17]. Tensor networks (TNs) provide a promising alternative for quantum RC by enabling the handling of complex quantum correlations and entanglement [18]–[30]. By introducing randomized effects into TNs, akin to the randomization in traditional RC, diverse quantum correlation patterns are generated, resulting in robust and generalizable quantum dynamic models [31]–[41].

B. Our results

In this study, we explore the TNRC approach for both static data classification and time-series dynamic predictions, with the following key findings:

1) The proposed TNRC method is computationally efficient, avoiding the high computational costs typical of traditional TN training.
2) Numerical calculations of IPC for time-series data show scaling behavior near the edge of chaos, resembling patterns observed in RC settings for digit classification.
3) Calculations indicate a universal normalized entanglement entropy predicting phase boundaries across various cases. Our research bridges the gap between traditional RC frameworks and quantum dynamics, utilizing randomized tensor networks to offer new insights into learnability and phase transitions within complex quantum systems.

Our research bridges the gap between traditional RC frameworks and quantum dynamics, utilizing randomized tensor networks to offer new insights into learnability and phase transitions within complex quantum systems.

II. Methods

A. Summary of Learning Architecture

The training dataset is denoted as $D = \left\{(\vec{x}^{[n]}, \vec{y}^{[n]})\right\}_{n=1}^{m}$ where each input vector $\vec{x}^{[n]} = \left(x_1^{[n]}, x_2^{[n]}, \ldots, x_N^{[n]}\right) \in [0,1]^N \subset \mathbb{R}^N$ and m represents the dataset size, while N is the input dimension. The output $\vec{y} = (y_1, y_2, \ldots, y_{|c|})$ is one-hot encoded for classification, determined as: $y_i = \delta_{i,k},\ k = \underset{j\in c}{\operatorname{argmax}} \sum_{l=0}^{L-1} W_{j,l} F_l(\vec{x})$ with $F_l(\vec{x}) = \sigma(f^{(l)}(\vec{x}))$, where $\sigma(z) = \frac{1}{1+e^{-z}}$ is the logistic sigmoid, and W is the trainable weight matrix.

The model utilizes a hybrid tensor-neural network with tensor-based decision functions and feature maps. For image input, the scikit-learn hand-written digit dataset [42]-[43] (8x8 grayscale images) is serialized, resulting in $N = 64$ and MPS reservoir length $N + 1 = 65$. The class set for output is $c = \{0, 1, \ldots, 9\}$, and a quadratic loss is used for training, with W obtained via Moore-Penrose pseudoinverse.

Bulletin of Networking, Computing, Systems, and Software – www.bncss.org, ISSN 2186-5140
Volume 14, Number 1, pages 77–80, January 2025

The decision function $f^{(l)}(\vec{x})$ is expressed as:

$$f^{(l)}(\vec{x}) = \sum_{i_1,i_2,\ldots,i_N=0}^{1} \Psi^{(l)}_{i_1,i_2,\ldots,i_N} \phi_{i_1}(x_1)\phi_{i_2}(x_2)\cdots\phi_{i_N}(x_N) \tag{1}$$

where Ψ is approximated by a Matrix Product State (MPS) as:

$$\begin{aligned}\Psi^{(l)}_{i_1,i_2,\ldots,i_N} &= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_N=0}^{\chi-1} A^{(1)}_{i_1,0,\alpha_1} A^{(2)}_{i_2,\alpha_1,\alpha_2} \ldots A^{(\frac{N}{2})}_{i_{(\frac{N}{2})},\alpha_{(\frac{N}{2}-1)},\alpha_{(\frac{N}{2})}} \\ &\times A^{(out)}_{l,\alpha_{(\frac{N}{2})},\alpha_{(\frac{N}{2}+1)}} A^{(\frac{N}{2}+1)}_{i_{(\frac{N}{2}+1)},\alpha_{(\frac{N}{2}+1)},\alpha_{(\frac{N}{2}+2)}} \ldots A^{(N)}_{i_N,\alpha_N,0}\end{aligned} \tag{2}$$

Here, χ denotes the bond dimension, and A matrices are randomly initialized from a Gaussian ensemble. MPS length is $N+1$, and basis functions for input encoding are $\phi_0(x) = 1 - x$ and $\phi_1(x) = x$. The MPS representation reduces the parameter count to $O(N\chi^2 L)$, making it an efficient compression for high-rank tensors, as opposed to the full representation of $2^N L$. During training, the MPS remains fixed. Entanglement entropy S is calculated for the MPS in mixed canonical form, normalized as $\widetilde{S} = S/\log_2\chi \in [0,\ 1]$. $\widetilde{S} = 1$ implies maximal entanglement, and $\widetilde{S} = 0$ indicates no entanglement.

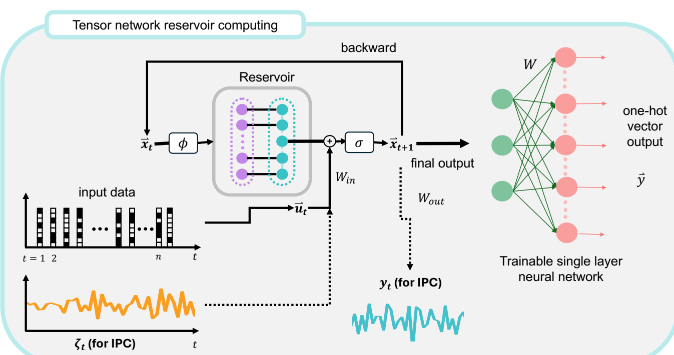

Fig. 1. The data flow from input to output of TNRC.

B. *Reservoir Computing and Information Processing Capacity (IPC)*

In this approach using reservoir computing for sequential data learning, each 8×8 image is divided into four sections, each containing 16 pixels. This transforms the image into four time slices, with each slice providing a 16-dimensional input vector $u_j(t)$ at time $t \in \{1,2,3,4\}$. The iterative function is defined as follows: $T_l(x_t,\ \zeta_t) = \sigma(f^{(l)}(\vec{x}_t) + \iota\sum_{j=1}^{M} W_{in,l,j}u_j(t))$. The final output is determined by: $y = \underset{i\in c}{\operatorname{argmax}} \sum_{l=1}^{N} W_{out,i,l}\sigma(f^{(l)}(\vec{x}_t) + \iota\sum_{j=1}^{M} W_{in,l,j}u_j(t))$ where $M = 16$, $N = 64$, and $t = 4$. Information Processing Capacity (IPC) is a measure based on memory capacity, evaluating how effectively the reservoir can model time-series data [44]. A system in a stable or chaotic state has low IPC, while a system at the edge of chaos has high IPC. For IPC calculation, we used the code provided by [45]. The time series is calculated using the transition function $T : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}^N$ as follows: $x_{t+1} = \vec{T}(\vec{x}_t, \zeta_t)$ for uniformly distributed random input signal $\zeta_t \in [-1,1]$. The transition function for tensor network-based reservoir computing is given by: $T_l(\vec{x}_t, \zeta_t) = \sigma(f^{(l)}(\vec{x}_t) + \iota\ W_{in,l}\zeta_t)$ where the input weight vector W_{in} is randomly drawn from the interval $[-1,1]$, and a scaling constant $\iota = 0.1$ is set. This flow is shown in Fig. 1.

III. Results and Discussions

A. *IPC and Dynamic Data*

The IPC analysis for time-series data demonstrates the transition behavior as the system approaches the chaotic phase. Using $N = 64$ for IPC calculation, Fig 2(a) shows that IPC starts at a low value in the stable phase, rises to its theoretical maximum $C_{tot} = N$ near the edge of chaos, and declines sharply as the system enters chaos. This phase transition highlights the system's stability and information-processing capacity. Fig. 2(b) provides a detailed phase diagram, showing that IPC reaches optimal values near the edge of chaos, reinforcing the framework's robustness across various datasets. Additionally, Fig. 2(c) illustrates the phase transition using handwritten digit data, confirming the adaptability of the IPC framework across different input dimensions and datasets.

B. *Entanglement Entropy of Randomized MPS Reservoirs*

The entanglement entropy of the randomized MPS reservoir is examined to understand its relationship with model performance. As the standard deviation σ_A of the Gaussian ensemble increases, the entanglement entropy also increases. Fig. 3(a) shows IPC plotted against the entanglement entropy, with phase transition points occurring consistently at the same entropy level across bond dimensions. Additionally, Fig. 3(b) reveals that the normalized entanglement entropy and test accuracy is observed for RC learning, with optimal fitting seen within $\widetilde{S} \in [0.1, 0.5]$. The results suggest that higher entanglement does not necessarily improve learning performance, aligning with known findings in quantum information theory [46]-[47].

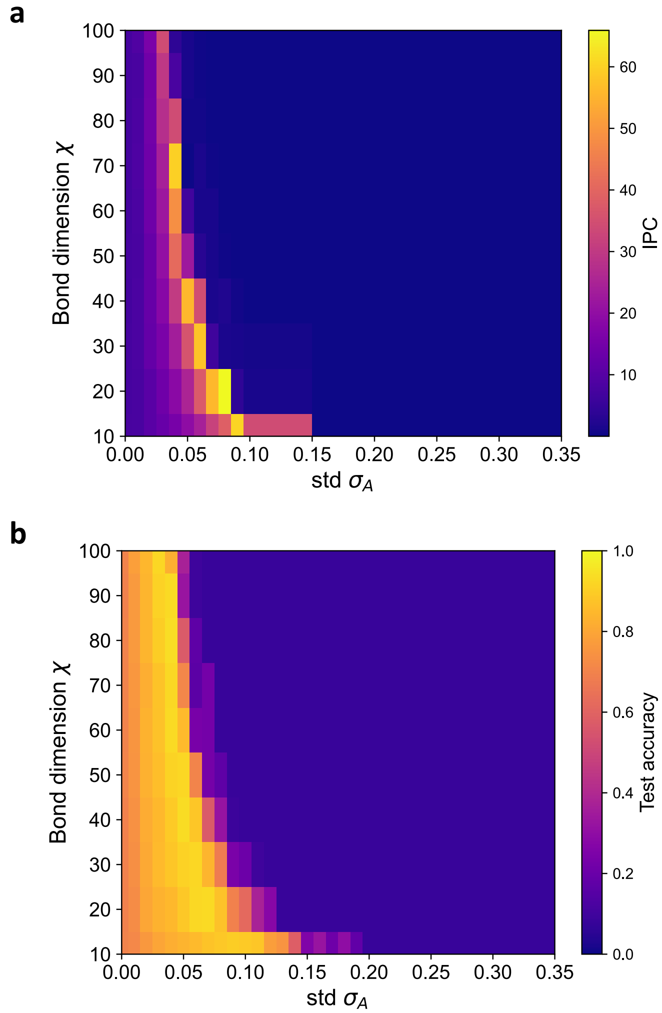

Fig. 2. The IPC for 64-dimensional input-output time series and RC learning data. (a) The IPC phase diagram (b) The phase diagram for the test accuracy of RC learning.

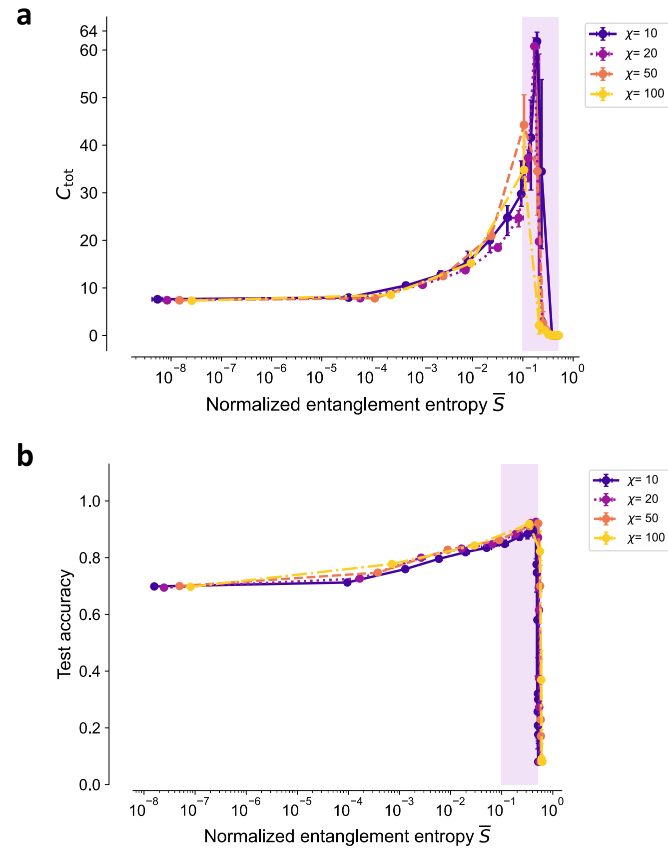

Fig. 3. (a) Test accuracy with respect to the normalized entanglement entropy for various bond dimensions. (b) The total IPC is plotted with respect to the normalized entanglement entropy $\widetilde{S}$

IV. Conclusion

This article introduces and demonstrates the TNRC method, validating its low-cost learning approach through a handwritten digit classification experiment. Empirical and theoretical analyses reveal learnability phase transitions in the random tensor reservoir, moving from underfitting to overfitting. The order-to-chaos criticality in reservoir computing (RC) is numerically evaluated using IPC and observed in time-sequence data experiments, with theoretical insights explaining these transitions. The entanglement entropy appears to play a role in these phase transitions, and further investigation involving higher moments may provide deeper insights.

Future work could explore TNRC applications in higher-dimensional tensor networks, neural network reservoirs, or other physical reservoirs. Applying TNRC to additional time-series settings and examining why critical entanglement entropy seems to be a universal constant remain open questions. We plan to conduct further analyses on practical time-series data and develop theoretical boundaries.

References

[1] H. Jaeger, "The "echo state" approach to analysing and training recurrent neural networks-with an erratum note," *Bonn, Germany: German National Research Center for Information Technology GMD Technical Report*, vol. 148, no. 34, p. 13, 2001.

[2] W. Maass, T. Natschläger, and H. Markram, "Real-time computing without stable states: A new framework for neural computation based on perturbations," *Neural computation*, vol. 14, no. 11, pp. 2531–2560, 2002.

[3] H. Jaeger and H. Haas, "Harnessing nonlinearity: Predicting chaotic systems and saving energy in wireless communication," *science*, vol. 304, no. 5667, pp. 78–80, 2004.

[4] K. Nakajima, "Physical reservoir computing—an introductory perspective," *Japanese Journal of Applied Physics*, vol. 59, no. 6, p. 060501, 2020.

[5] F. Rosenblatt, "Principles of neurodynamics: Perceptrons and the theory of brain mechanisms," 1961.

[6] S.-I. Amari, "Learning patterns and pattern sequences by self-organizing nets of threshold elements," *IEEE Transactions on computers*, vol. 100, no. 11, pp. 1197–1206, 1972.

[7] G.-B. Huang, Q.-Y. Zhu, and C.-K. Siew, "Extreme learning machine: a new learning scheme of feedforward neural networks," in *2004 IEEE international joint conference on neural networks (IEEE Cat. No. 04CH37541)*, vol. 2. Ieee, 2004, pp. 985–990.

[8] M. Lukoševičius and H. Jaeger, "Reservoir computing approaches to recurrent neural network training," *Computer science review*, vol. 3, no. 3, pp. 127–149, 2009.

[9] G. Tanaka, T. Yamane, J. B. Héroux, R. Nakane, N. Kanazawa, S. Takeda, H. Numata, D. Nakano, and A. Hirose, "Recent advances in physical reservoir computing: A review," *Neural Networks*, vol. 115, pp. 100–123, 2019.

[10] S. Boyd and L. Chua, "Fading memory and the problem of approximating nonlinear operators with volterra series," *IEEE Transactions on circuits and systems*, vol. 32, no. 11, pp. 1150–1161, 1985.

[11] D. J. Gauthier, E. Bollt, A. Griffith, and W. A. Barbosa, "Next generation reservoir computing," *Nature communications*, vol. 12, no. 1, pp. 1–8, 2021.

[12] M. Cucchi, S. Abreu, G. Ciccone, D. Brunner, and H. Kleemann, "Hands-on reservoir computing: a tutorial for practical implementation," *Neuromorphic Computing and Engineering*, vol. 2, no. 3, p. 032002, 2022.

[13] M. Yan, C. Huang, P. Bienstman, P. Tino, W. Lin, and J. Sun, "Emerging opportunities and challenges for the future of reservoir computing," *Nature Communications*, vol. 15, no. 1, p. 2056, 2024.

[14] G. Passetti, D. Hofmann, P. Neitemeier, L. Grunwald, M. A. Sentef, and D. M. Kennes, "Can neural quantum states learn volume-law ground states?" *Physical Review Letters*, vol. 131, no. 3, p. 036502, 2023.

[15] G. Carleo, I. Cirac, K. Cranmer, L. Daudet, M. Schuld, N. Tishby, L. Vogt-Maranto, and L. Zdeborová, "Machine learning and the physical sciences," *Reviews of Modern Physics*, vol. 91, no. 4, p. 045002, 2019.

[16] T. Kimura, K. Shiba, C.-C. Chen, M. Sogabe, K. Sakamoto, and T. Sogabe, "Quantum circuit architectures via quantum observable markov decision process planning," *Journal of Physics Communications*, vol. 6, no. 7, p. 075006, 2022.

[17] C.-C. Chen, S.-Y. Shiau, M.-F. Wu, and Y.-R. Wu, "Hybrid classical-quantum linear solver using noisy intermediate-scale quantum machines," *Scientific reports*, vol. 9, no. 1, p. 16251, 2019.

[18] R. Orús, "Tensor networks for complex quantum systems," *Nature Reviews Physics*, vol. 1, no. 9, pp. 538–550, 2019.

[19] S. R. White, "Density matrix formulation for quantum renormalization groups," *Physical review letters*, vol. 69, no. 19, p. 2863, 1992.

[20] T. Nishino and K. Okunishi, "Corner transfer matrix renormalization group method," *Journal of the Physical Society of Japan*, vol. 65, no. 4, pp. 891–894, 1996.

[21] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, "Criticality, the area law, and the computational power of projected entangled pair states," *Physical review letters*, vol. 96, no. 22, p. 220601, 2006.

[22] G. Vidal, "Entanglement renormalization," *Physical review letters*, vol. 99, no. 22, p. 220405, 2007.

[23] M. P. Zaletel and F. Pollmann, "Isometric tensor network states in two dimensions," *Physical review letters*, vol. 124, no. 3, p. 037201, 2020.

[24] E. Jeckelmann, "Dynamical density-matrix renormalization-group method," *Physical Review B*, vol. 66, no. 4, p. 045114, 2002.

[25] I. V. Oseledets, "Tensor-train decomposition," *SIAM Journal on Scientific Computing*, vol. 33, no. 5, pp. 2295–2317, 2011.

[26] J. Eisert, M. Cramer, and M. B. Plenio, "Colloquium: Area laws for the entanglement entropy," *Reviews of modern physics*, vol. 82, no. 1, pp. 277–306, 2010.

[27] N. Schuch, M. M. Wolf, F. Verstraete, and J. I. Cirac, "Entropy scaling and simulability by matrix product states," *Physical review letters*, vol. 100, no. 3, p. 030504, 2008.

[28] S. Östlund and S. Rommer, "Thermodynamic limit of density matrix renormalization," *Physical review letters*, vol. 75, no. 19, p. 3537, 1995.

[29] U. Schollwöck, "The density-matrix renormalization group in the age of matrix product states," *Annals of physics*, vol. 326, no. 1, pp. 96–192, 2011.

[30] J. Dukelsky, M. A. Martín-Delgado, T. Nishino, and G. Sierra, "Equivalence of the variational matrix product method and the density matrix renormalization group applied to spin chains," *Europhysics letters*, vol. 43, no. 4, p. 457, 1998.

[31] J. M. Koh, S.-N. Sun, M. Motta, and A. J. Minnich, "Measurement-induced entanglement phase transition on a superconducting quantum processor with mid-circuit readout," *Nature Physics*, vol. 19, no. 9, pp. 1314–1319, 2023.

[32] S. González-García, S. Sang, T. H. Hsieh, S. Boixo, G. Vidal, A. C. Potter, and R. Vasseur, "Random insights into the complexity of two-dimensional tensor network calculations," *Physical Review B*, vol. 109, no. 23, p. 235102, 2024.

[33] N. Gourianov, M. Lubasch, S. Dolgov, Q. Y. van den Berg, H. Babaee, P. Givi, M. Kiffner, and D. Jaksch, "A quantum-inspired approach to exploit turbulence structures," *Nature Computational Science*, vol. 2, no. 1, pp. 30–37, 2022.

[34] F. Barratt, U. Agrawal, A. C. Potter, S. Gopalakrishnan, and R. Vasseur, "Transitions in the learnability of global charges from local measurements," *Physical review letters*, vol. 129, no. 20, p. 200602, 2022.

[35] J. Haferkamp, C. Bertoni, I. Roth, and J. Eisert, "Emergent statistical mechanics from properties of disordered random matrix product states," *PRX Quantum*, vol. 2, no. 4, p. 040308, 2021.

[36] J. Biamonte, M. Faccin, and M. De Domenico, "Complex networks from classical to quantum," *Communications Physics*, vol. 2, no. 1, p. 53, 2019.

[37] S. Morita, R. Igarashi, H.-H. Zhao, and N. Kawashima, "Tensor renormalization group with randomized singular value decomposition," *Physical Review E*, vol. 97, no. 3, p. 033310, 2018.

[38] B. Collins and I. Nechita, "Random matrix techniques in quantum information theory," *Journal of Mathematical Physics*, vol. 57, no. 1, 2016.

[39] B. Collins, C. E. González-Guillén, and D. Pérez-García, "Matrix product states, random matrix theory and the principle of maximum entropy," *Communications in Mathematical Physics*, vol. 320, no. 3, pp. 663–677, 2013.

[40] S. Garnerone, T. R. de Oliveira, and P. Zanardi, "Typicality in random matrix product states," *Physical Review A—Atomic, Molecular, and Optical Physics*, vol. 81, no. 3, p. 032336, 2010.

[41] M. M. Wolf, G. Ortiz, F. Verstraete, and J. I. Cirac, "Quantum phase transitions in matrix product systems," *Physical review letters*, vol. 97, no. 11, p. 110403, 2006.

[42] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay, "Scikit-learn: Machine learning in Python," *Journal of Machine Learning Research*, vol. 12, pp. 2825–2830, 2011.

[43] E. Alpaydin and C. Kaynak, "Optical Recognition of Handwritten Digits," UCI Machine Learning Repository, 1998, DOI: https://doi.org/10.24432/C50P49.

[44] J. Dambre, D. Verstraeten, B. Schrauwen, and S. Massar, "Information processing capacity of dynamical systems," *Scientific reports*, vol. 2, no. 1, p. 514, 2012.

[45] T. Kubota, H. Takahashi, and K. Nakajima, "Unifying framework for information processing in stochastically driven dynamical systems," *Phys. Rev. Res.*, vol. 3, p. 043135, Nov 2021. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevResearch.3.043135

[46] D. Gross, S. T. Flammia, and J. Eisert, "Most quantum states are too entangled to be useful as computational resources," *Physical review letters*, vol. 102, no. 19, p. 190501, 2009.

[47] S. Aaronson, "How much structure is needed for huge quantum speedups?" 2022. [Online]. Available: https://arxiv.org/abs/2209.06930